Input-to-state stability of interconnected hybrid systems ⋆

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Abstract

We consider the interconnections of arbitrary topology of a finite number of ISS hybrid systems and study whether the ISS property is maintained for the overall system. We show that if the small gain condition is satisfied, then the whole network is ISS and show how a non-smooth ISS-Lyapunov function can be explicitly constructed in this case.

Key words: Stability of hybrid systems; Lyapunov methods; Large-scale systems.

1 Introduction

Hybrid systems allow for a combination of continuous and discontinuous types of behavior in one model and hence can be used in many applications, for example in robotics [1], reset systems [19] or networked control systems [23,18]. Such systems often have a large scale interconnected structure and can be naturally modeled as interconnected hybrid systems. In this paper our main interest is in stability and robustness for such interconnections as these properties are certainly of great importance for applications. We will use the framework of input-to-state stability (ISS) that was first introduced for continuous systems in [22] and then extended to other types of systems including hybrid ones, see for example [2], [10], [11] and [14]. The ISS property of the interconnected systems is usually studied using small gain conditions that take the interconnection structure of the whole system into account. First small gain conditions for ISS systems were introduced for the interconnections of two continuous systems in [13,12]. These results were extended to arbitrary number of interconnected systems in [5,6,16].
Interconnections of two hybrid ISS systems were considered in [17], [18], e.g.. A stability condition of the small gain type was used in [18] for a construction of an ISS-Lyapunov function for their feedback connection. Interconnection of arbitrary number of sampled-data systems that are a special class of hybrid systems was considered in [16]. The small gain condition was given there in terms of vector Lyapunov functions.

In this paper we obtain similar results for the interconnection of arbitrary number of hybrid systems. To this end we use the methodology recently developed in [5], [6] for the investigation of stability of general networks of ISS systems. In particular we use the small gain condition developed in these papers and we use non-smooth ISS-Lyapunov functions in our considerations. The main result of this paper extends the result of [18] for the case of interconnection of $n \geq 2$ hybrid systems and [16] for general type of hybrid systems by applying the small gain condition in the matrix form. Moreover, we prove the small gain results in terms of trajectories. There are different ways to introduce ISS-Lyapunov functions for hybrid systems, see for example [3], [18]. We show their equivalence in this paper. Using the methods developed in [6] we provide an explicit construction of an ISS-Lyapunov function for interconnected hybrid systems.
The next section introduces all necessary notions and notation. Section 3 contains the main results and Section 4 concludes the paper.

2 Preliminaries

Let $\mathbb{R}_+$ be the set of nonnegative real numbers, $\mathbb{R}^n_+$ be the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{N}_+ := \{0, 1, 2, \ldots\}$. x^T stands for the transposition of a vector $x \in \mathbb{R}^n$. $\mathbb{B}$ is the open unit ball centered at the origin in $\mathbb{R}^n$ and $\bar{\mathbb{B}}$ is its closure. Set $B \subset \mathbb{R}^n$ is called *relatively closed in* $\chi \subset \mathbb{R}^n$, if $B = \bar{B} \cap \chi$. By $\langle \cdot, \cdot \rangle$ we denote the standard scalar product in $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we write $x \geq y \Leftrightarrow x_i \geq y_i$; $x > y \Leftrightarrow x_i > y_i, i = 1, \ldots, n$; $x \ngeq y \Leftrightarrow \exists i \in \{1, \ldots, n\} : x_i < y_i$. M^n denotes the n-fold composition $M \circ \ldots \circ M$ of a map $M : \mathbb{R}^n_+ \to \mathbb{R}^n_+$.
A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$ and $\alpha(t) > 0$ for $t > 0$ is called positive definite. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly in-

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creasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. Note that for $\gamma \in \mathcal{K}_\infty$ the inverse function $\gamma^{-1} \in \mathcal{K}_\infty$ always exists. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero for $t \to \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KLL}$ if, for each fixed $r \geq 0$, $\beta(\cdot, \cdot, r) \in \mathcal{KL}$ and $\beta(\cdot, r, \cdot) \in \mathcal{KL}$.

2.1 Interconnected hybrid systems

Consider an interconnection of n hybrid subsystems with states $x_i \in \chi_i \subset \mathbb{R}^{N_i}$, $i=1,\ldots,n$, and external input $u \in U \subset \mathbb{R}^M$. Dynamics of the ith subsystem is given by

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \ldots, x_n, u), (x_1, \ldots, x_n, u) \in C_i \\ x_i^+ &= g_i(x_1, \ldots, x_n, u), (x_1, \ldots, x_n, u) \in D_i \end{aligned} \tag{1}$$

where $f_i : C_i \to \mathbb{R}^{N_i}, g_i : D_i \to \chi_i$ and C_i, D_i are the subsets of $\chi_1 \times \ldots \times \chi_n \times U$. Each subsystem is described by $(f_i, g_i, C_i, D_i, \chi_i, U)$, however in view of stability properties we will need to restrict such interconnections to $D_i = D_j$, $\forall i, j$, see Remark 2.1 below.

If $(x_1, \ldots, x_n, u) \in C_i$, then system (1) flows continuously and the dynamics is given by function f_i. If $(x_1, \ldots, x_n, u) \in D_i$, then the system jumps instantaneously according to function g_i. In points of $C_i \cap D_i$ the system may either flow or jump, the latter only if the flowing keeps $(x_1, \ldots, x_n, u) \in C_i$. Define $\chi := \chi_1 \times \ldots \times \chi_n$. The solutions are defined on hybrid time domains [7]. A subset $\mathbb{R}_+ \times \mathbb{N}_+$ is called hybrid time domain denoted by *dom* if it is given as a union of finitely or infinitely many intervals $[t_k, t_{k+1}] \times \{k\}$, where the numbers $0 = t_0, t_1, \ldots$ form a finite or infinite, nondecreasing sequence of real numbers. The "last" interval is allowed to be of the form $[t_K, T) \times \{k\}$ with T finite or $T = +\infty$.

A hybrid signal is a function defined on the hybrid time domain. For the ith subsystem the hybrid input

$$v_i := (x_1^T, \ldots, x_{i-1}^T, x_{i+1}^T, \ldots, x_n^T, u^T)^T, \tag{2}$$

consists of hybrid signals u : dom $u \to U \subset \mathbb{R}^M$, $x_j : \operatorname{dom} x_j \to \chi_j, j \neq i$ such that $u(\cdot, k), x_j(\cdot, k)$ are Lebesgue measurable and locally essentially bounded for each k. For a signal $u{:}\operatorname{dom} u \to U \subset \mathbb{R}^M$ we define its restriction to the interval $[(t_1, j_1), (t_2, j_2)] \in \operatorname{dom} u$ by

$$u_{[(t_1,j_1),(t_2,j_2)]}(t,k) = \begin{cases} u(t,k), & \text{if } (t_1, j_1) \leq (t,k) \leq (t_2, j_2), \\ 0, & \text{otherwise,} \end{cases}$$

where for the elements of the hybrid time domain we define that $(s, l) \leq (t, k)$ means $s + l \leq t + k$. For convenience, we denote $u_{(t,k)} := u_{[(0,0),(t,k)]}$.

A hybrid arc of subsystem i is such a hybrid signal $x_i : \operatorname{dom} x_i \to \chi_i$, that $x_i(\cdot, k)$ is locally absolutely continuous for each k. Define $x := (x_1^T, \ldots, x_n^T)^T \in \chi \subset \mathbb{R}^N$, $N := \sum N_i$. A hybrid arc and a hybrid input is a solution pair (x_i, v_i) of the ith hybrid subsystem (1) if

(i) $\operatorname{dom} x_i = \operatorname{dom} u = \operatorname{dom} x_j, j \neq i$ and

$$(x(0,0), u(0,0)) \in C_i \cup D_i,$$

(ii) for all $k \in N_+$ and almost all $(t, k) \in$ dom x_i

$$\dot{x}_i(t,k) = f_i(x(t,k), u(t,k)), \text{ if } (x(t,k), u(t,k)) \in C_i \tag{3}$$

(iii) for all $(t, k) \in \operatorname{dom} x_i$ such that $(t, k+1) \in \operatorname{dom} x_i$

$$x_i(t,k+1) = g_i(x(t,k), u(t,k)), \text{ if } (x(t,k), u(t,k)) \in D_i. \tag{4}$$

For the existence of solutions assume that the following basic regularity conditions [3], [8] hold :

(1) χ_i is open, U is closed, and $C_i, D_i \subset \chi \times U$ are relatively closed in $\chi \times U$;
(2) f_i, g_i are continuous.

The supremum norm of a hybrid signal u defined on $[(0, 0), (t, k)] \in \operatorname{dom} u$ is defined by

$$\|u\|_{(t,k)} := \max \left\{ \operatorname*{ess\,sup}_{\substack{(s,l) \in \operatorname{dom} u \setminus \Phi(u), \\ (s,l) \leq (t,k)}} |u(s,l)|, \sup_{\substack{(s,l) \in \Phi(u), \\ (s,l) \leq (t,k)}} |u(s,l)| \right\}$$

and $\Phi(u) := \{(s,l) \in \operatorname{dom} : (s, l+1) \in \operatorname{dom} u\}$. If $t+k \to \infty$, then $\|u\|_{(t,k)}$ is denoted by $\|u\|_\infty$. The set of hybrid inputs in $\mathbb{R}^M$ with finite $\|\cdot\|_\infty$ is denoted by $\mathcal{L}_\infty^M$. A solution pair of hybrid system is *maximal* if it cannot be extended. It is *complete* if its hybrid time domain is unbounded. Let $S_u(x_0)$ be the set of all maximal solution pairs (x, u) to (5) with $x(0, 0) = x_0$.

To consider interconnection (1) as one hybrid system

$$\begin{aligned} \dot{x} &= f(x, u), \ (x, u) \in C, \\ x^+ &= g(x, u), \ (x, u) \in D, \end{aligned} \tag{5}$$

with state x and input u defined above, it seems to be natural to define $C := \cap C_i$, $D := \cup D_i$, since a jump of any subsystem means a jump for the overall state x, and to define function $f : C \to \mathbb{R}^N$ by $f := (f_1^T, \ldots, f_n^T)^T$ and function $g : D \to \chi$ as follows $g := (\widetilde{g}_1^T, \ldots, \widetilde{g}_n^T)^T$, where

$$\widetilde{g}_i(x, u) := \begin{cases} g_i(x, u), & \text{if } (x, u) \in D_i, \\ x_i, & \text{otherwise .} \end{cases} \tag{6}$$

Note that the solutions of (5) may have different hybrid time domains than the solutions of the individual systems (1), see [21]. The above choice of C and D was used also in [21] considering interconnections of two hybrid systems. However this choice has certain drawbacks: (6) rules out solutions starting in $C \cap D$ such that one subsystem jumps while another one flows, another problem is discussed in Remark 2.1, see also [21, Remark 4.3].

Remark 2.1 *Let one of the subsystems, say the jth one, has the property that once being in D_j it makes only jumps and never leaves D_j. If $D_i \neq D_j$ for some i, then for any initial state $x(0,0)$ with $(x(0,0), u(0,0)) \in D_j$ and $(x(0,0), u(0,0)) \in C_i \setminus D_i$ there exists a solution pair given by $(x(0,k), u(0,k))$, $k \in \mathbb{N}$ with $x_i(0,k) = x_i(0,0)$, $\forall k$, i.e., a solution with the "frozen" x_i. This follows from (6): being in C_i we have $\tilde{g}_i = id$. This in particular shows that even in case of a zero input signal there is a solution that will never become "small", contradicting the ISS or the AG property (defined below). For this reason we require in Section 3 that the jump sets D_i coincide for all subsystems. This requirement implies that the subsystems can*

jump simultaneously only. This restricts the class of interconnected systems considered in this paper.

2.2 Input-to-state stability and Lyapunov functions

To study stability of the interconnected hybrid systems we use the notion of input-to-state stability (ISS) [3]:

Definition 2.2 *The ith subsystem (1) is called ISS, if there exist $\beta_i \in \mathcal{KLL}$, γ_{ij}, $\gamma_i \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x_{i0} each solution pair $(x_i, v_i) \in S_{v_i}(x_{i0})$ with v_i from (2) satisfies $\forall (t,k) \in dom\, x_i$ the following:*

$$|x_i(t,k)| \leq \max\{\beta_i(|x_{i0}|,t,k), \max_{j,j\neq i} \gamma_{ij}(\|x_j\|_{(t,k)}), \gamma_i(\|u\|_{(t,k)})\}. \tag{7}$$

Functions γ_{ij}, γ_i are called ISS nonlinear gains.

We borrow also the following stability notions from [3] that will be used in the next section to prove one of the main results (Theorem 3.4):

Definition 2.3 *System (1) is called* 0*-input pre-stable, if for each $\epsilon_i > 0$ there exists $\delta_i > 0$ such that each solution pair $(x_i, 0) \in S_{v_i}(x_{i0})$ with $|x_{i0}| \leq \delta$ satisfies $|x_i(t,k)| \leq \epsilon_i$ for all $(t,k) \in dom\, x_i$.*

Definition 2.4 *System (1) is called globally pre-stable (pre-GS), if $\exists \sigma_i, \hat{\gamma}_{ij}, \hat{\gamma}_i \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x_{i0} each solution pair $(x_i, v_i) \in S_{v_i}(x_{i0})$ satisfies $\forall (t,k) \in dom\, x_i$ the following:*

$$|x_i(t,k)| \leq \max\{\sigma_i(|x_{i0}|), \max_{j,j\neq i} \hat{\gamma}_{ij}(\|x_j\|_{(t,k)}), \hat{\gamma}_i(\|u\|_{(t,k)})\} \tag{8}$$

Remark 2.5 *Note that pre-GS follows from ISS by taking $\sigma_i(|x_{i0}|) := \beta_i(|x_{i0}|, 0, 0)$ and* 0*-input pre-stability follows from pre-GS by considering $x_j = 0, u = 0$.*

Definition 2.6 *System (1) has the asymptotic gain property (AG), if there exist $\widetilde{\gamma}_{ij}, \widetilde{\gamma}_i \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x_{i0} all solution pairs $(x_i, v_i) \in S_{v_i}(x_{i0})$ are bounded and, if complete, then satisfy*

$$\limsup_{\substack{(t,k)\in dom\, x,\\ t+k\to\infty}} |x_i(t,k)| \leq \max\{\max_{j,j\neq i} \widetilde{\gamma}_{ij}(\|x_j\|_\infty), \widetilde{\gamma}_i(\|u\|_\infty)\}. \tag{9}$$

In Theorem 3.1 in [3] the following relation between ISS and AG with 0-input pre-stability was proved.

Theorem 2.7 *Let the set $\{f_i(x,u) : u \in U \cap \epsilon\bar{\mathbb{B}}\}$ be convex $\forall \varepsilon > 0$ and for any $x \in \chi$. Then (1) is ISS if and only if it has the AG property and it is* 0*-input pre-stable.*

A common alternative to prove ISS is to use ISS-Lyapunov functions as defined below. We consider locally Lipschitz continuous functions $V_i : \chi_i \to \mathbb{R}_+$ that are differentiable almost everywhere by the Rademacher's theorem. The set of such functions we denote by Lip_{loc}. In points where such a function is not differentiable we use the notion of Clarke's generalized gradient, see [4], [6]. The set

$$\partial V_i(x_i) = \mathrm{conv}\{\zeta_i \in \mathbb{R}^{n_i} : \exists x_i^p \to x_i, \exists \nabla V_i(x_i^p) \text{ and } \nabla V_i(x_i^p) \to \zeta_i\} \tag{10}$$

is called *Clarke's generalized gradient of V_i at $x_i \in \chi_i$*. If V_i is differentiable at some point, then $\partial V_i(x_i)$ coincides with the usual gradient at this point.

Definition 2.8 *Function $V_i : \chi_i \to \mathbb{R}_+$, $V_i \in Lip_{loc}$ is called an ISS-Lyapunov function for system (1) if*
1) There exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ s.t.:

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|) \text{ for any } x_i \in \chi_i. \tag{11}$$

2) There exist continuous, proper, positive definite functions $V_j : \chi_j \to \mathbb{R}$, $V_j \in Lip_{loc}$, $j \in \{1, \ldots, n\} \setminus \{i\}$, functions $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty$ and continuous, positive definite functions α_i, λ_i, with $\lambda_i(s) < s$ for all $s > 0$ such that for all $(x,u) \in C_i$

$$\begin{aligned} &V_i(x_i) \geq \max\{\max_{j,j\neq i}\{\gamma_{ij}(V_j(x_j))\}, \gamma_i(|u|)\} \Rightarrow \\ &\forall \zeta_i \in \partial V_i(x_i) : \langle \zeta_i, f_i(x,u) \rangle \leq -\alpha_i(V_i(x_i)) \end{aligned} \tag{12}$$

and for all $(x,u) \in D_i$

$$V_i(g_i(x,u)) \leq \max\{\lambda_i(V_i(x_i)), \max_{j,j\neq i}\{\gamma_{ij}(V_j(x_j))\}, \gamma_i(|u|)\}. \tag{13}$$

Functions γ_{ij}, γ_i are called ISS Lyapunov gains corresponding to the inputs x_j and u respectively.

Note that this definition is different from the definition of an ISS Lyapunov function used in [3]. The equivalence between their existence for (1) is shown in Appendix, Section A. Note also that γ_{ij} are taken the same in (12) and (13). This can be always achieved by taking the maximums of separately obtained γ_{ij}'s for the continuous and discrete dynamics. If V_i is differentiable at x_i, then (12) can be written as

$$\begin{aligned} &V_i(x_i) \geq \max\{\max_j\{\gamma_{ij}(V_j(x_j))\}, \gamma_i(|u|)\} \Rightarrow \\ &\nabla V_i(x_i) \cdot f_i(x,u) \leq -\alpha_i(V_i(x_i)), (x,u) \in C_i. \end{aligned}$$

Relations between the existence of a smooth ISS-Lyapunov function and the ISS property for hybrid systems were discussed in [3]. Proposition 2.7 in [3] shows that if a hybrid system has an ISS-Lyapunov function, then it is ISS. Example 3.4 in [3] shows that the converse is in general not true. In [3, Theorem 3.1] it was proved that if (5) is ISS with f such that the set $\{f(x,u) : u \in U \cap \epsilon\bar{\mathbb{B}}\}$ is convex $\forall \varepsilon > 0$ and for any $x \in \chi$, then it has an ISS-Lyapunov function. Usually Lyapunov function is required to be smooth, but smoothness can be relaxed to locally Lipschitzness as shown below.

Proposition 2.9 *If system (1) has a locally Lipschitz continuous ISS-Lyapunov function, then it is ISS.*

Sketch of proof. The proof of [3, Proposition 2.7] stated with $\alpha_i \in \mathcal{K}_\infty$ works without change if α_i is continuous and positive definite. As well this proof can be extended to the nonsmooth V_i using the Clarke's generalized derivative. The assertion of the proposition follows then from this extension and Proposition A.1 from Section A in Appendix. □

Note that ISS of all subsystems does no guarantee ISS of their interconnection [5]. In the following section we introduce conditions that guarantee stability for interconnections of ISS hybrid systems.

3 Main results

The main question of this paper is whether the interconnection (5) of the ISS subsystems (1) is ISS. To study

this question we collect the gains γ_{ij} of the subsystems in the matrix $\Gamma=(\gamma_{ij})_{n\times n}$, $i,j{=}1,\ldots,n$ denoting $\gamma_{ii}\equiv 0$, $i=1,\ldots,n$, for completeness [5,20]. The matrix Γ describes the interconnection topology of the whole network and contains the information about the mutual influence between the subsystems. We also introduce the following gain operator $\Gamma_{\max}:\mathbb{R}^n_+\to\mathbb{R}^n_+$, see [5,20,16]:

$$\Gamma_{\max}(s):=\begin{pmatrix}\max\{\gamma_{12}(s_2),\ldots,\gamma_{1n}(s_n)\}\\ \vdots\\ \max\{\gamma_{n1}(s_1),\ldots,\gamma_{n,n-1}(s_{n-1})\}\end{pmatrix}. \quad (14)$$

We define the small gain condition as follows:

$$\Gamma_{\max}(s)\not\geq s,\quad \forall s\in\mathbb{R}^n_+,\ s\neq 0. \quad (15)$$

This condition was introduced and studied in [5] and [20]. Furthermore, in [5] it was shown that (15) is equivalent to the so-called cycle condition [16]. We will see that condition (15) guarantees stability of the network.

3.1 Small gain theorems in terms of trajectories

The following small gain theorems extend Theorem 4.1 and Theorem 4.2 in [5] to the case of hybrid systems.

Theorem 3.1 *Consider system (5) and assume that all its subsystems are pre-GS. If $\Gamma_{\max}$ defined in (14) with $\gamma_{ij}=\hat{\gamma}_{ij}$ satisfies (15), then (5) is pre-GS, i.e. for some $\sigma,\hat{\gamma}\in\mathcal{K}_\infty\cup\{0\}$ and for all $(t,k)\in dom\,x$*

$$|x(t,k)|\leq\max\{\sigma(|x_0|),\hat{\gamma}(\|u\|_{(t,k)})\}. \quad (16)$$

Theorem 3.2 *Consider the interconnected system (5) with $D_i=D, i{=}1,\ldots,n$. Assume that each subsystem (1) has the AG property and that solutions of the system (5) exist, are bounded and some of them are complete. If $\Gamma_{\max}$ defined by (14) with $\gamma_{ij}=\widetilde{\gamma}_{ij}$ satisfies (15) then system (5) satisfies the AG property. In particular any complete solution with some $\widetilde{\gamma}\in\mathcal{K}_\infty\cup\{0\}$ satisfies*

$$\limsup_{(t,k)\in dom\,x,t+k\to\infty}|x(t,k)|\leq\widetilde{\gamma}(\|u\|_\infty). \quad (17)$$

Note that if all solutions of (5) are not complete, then (5) is AG by definition. See Appendix B.1, B.2 for the proofs of Theorem 3.1 and 3.2.

Remark 3.3 *The existence and boundedness of solutions of (5) is essential, otherwise the assertion is not true, see Example 14 in [5].*

The following theorem extends [17, Theorem 1] showing ISS of an interconnection of two ISS hybrid systems under the small gain condition. Here we show that the same holds for arbitrary finite number of hybrid systems.

Theorem 3.4 *Consider the interconnected system (5) with $D_i=D, i=1,\ldots,n$. Assume that the set $\{f(x,u): u\in U\cap\epsilon\bar{\mathbb{B}}\}$ is convex for each $x\in\chi,\epsilon>0$. If all subsystems in (1) are ISS and $\Gamma_{\max}$ defined in (14) satisfies (15), then (5) is ISS, i.e. for $(t,k){\in}dom\,x$*

$$|x(t,k)|{\leq}\max\{\beta(|x_0|,t,k),\gamma(\|u\|_{(t,k)})\} \quad (18)$$

holds for some $\beta\in\mathcal{KLL}$ and $\gamma\in\mathcal{K}_\infty\cup\{0\}$.

Sketch of proof. The idea follows from the proof of a similar theorem for continuous systems in [5]. We describe it briefly: By Remark 2.5 and Theorem 2.7, since each subsystem is ISS, they are pre-GS and have the AG property. By Theorem 3.1 and Theorem 3.2 the whole interconnection (5) is pre-GS and has the AG property. From global pre-stability of (5), 0-input pre-stability follows, see Remark 2.5. ISS of (5) follows then by Theorem 2.7. □

Remark 3.5 *In comparison to Theorem 1 in [17], we require additionally in Theorem 3.4 that the set $\{f(x,u){:}u{\in}U{\cap}\epsilon\bar{\mathbb{B}}\}$ is convex for each $x{\in}\chi,\epsilon{>}0$. This is due to the fact that we use in our proof that ISS is equivalent to 0-input pre-stability and the AG property. This equivalence requires that the set $\{f(x,u){:}u{\in}U{\cap}\epsilon\bar{\mathbb{B}}\}$ is convex for each $x{\in}\chi,\epsilon{>}0$, see [3]. However, we do not exclude that it might be possible to prove the theorem without this equivalence and to avoid this restriction.*

3.2 Small-gain theorems in terms of Lyapunov functions

In this section we show how an ISS Lyapunov function for an interconnection (5) can be constructed using the small gain condition. This allows to apply Proposition 2.9 to deduce ISS of (5).

Theorem 3.6 *Consider system (5) as interconnection of subsystems (1) with $D_i=D, i=1,\ldots,n$ and assume that each subsystem i has an ISS Lyapunov function V_i satisfying (11)-(13) with corresponding ISS-Lyapunov gains. Let the corresponding gain operator $\Gamma_{\max}$, defined by (14) in terms of these gains, satisfy (15), then the hybrid system (5) has an ISS-Lyapunov function given by*

$$V(x)=\max_i\sigma_i^{-1}(V_i(x_i)) \quad (19)$$

where $\sigma_i(r):=\max\{a_ir,(\Gamma_{\max}(ar))_i,\ldots,(\Gamma^{n-1}_{\max}(ar))_i\}$, $r{\in}\mathbb{R}_+$ with an arbitrary positive vector $a{=}(a_1,\ldots,a_n)^T$. In particular, function V satisfies:
1) There exist functions $\psi_1,\psi_2\in\mathcal{K}_\infty$ s.t.:

$$\psi_1(|x|)\leq V(x)\leq\psi_2(|x|)\ \textit{for any}\ x\in\chi. \quad (20)$$

2) There exist function $\gamma\in\mathcal{K}$, and continuous, positive definite functions α, λ with $\lambda(s)<s$ for all $s>0$ s.t.:

$$V(x){\geq}\gamma(|u|){\Rightarrow}\forall\zeta{\in}\partial V(x){:}\langle\zeta,f(x,u)\rangle{\leq}{-}\alpha(V(x)),(x,u){\in}C, \quad (21)$$

$$V(g(x,u)){\leq}\max\{\lambda(V(x),\gamma(|u|)\},(x,u){\in}D. \quad (22)$$

Proof. First, we establish some regularity and monotonicity properties of σ_i. Then we apply these properties to show that V constructed in (19) satisfies (20)-(22). Without loss of generality, the gains γ_{ij} can be assumed to be smooth on $(0,\infty)$, see [9, Lemma B.2.1]. To establish the properties of σ_i consider the map $Q{:}\mathbb{R}^n_+{\to}\mathbb{R}^n_+$ defined by $Q(x){:=}(Q_1(x),\ldots,Q_n(x))^T$ with $Q_i(s){:=}\max\{s_i,(\Gamma_{\max}(s))_i,\ldots,(\Gamma^{n-1}_{\max}(s))_i\}$. Note that

according to the notation given in Section 2, $(\Gamma^p_{\max}(s))_i$ is the ith component of the vector $\Gamma^p_{\max}(s)$ where $\Gamma^p_{\max}$ is the p-fold composition of $\Gamma_{\max}$. By [16, Proposition 2.7] the inequality $\Gamma_{\max}(Q(x)){\le}Q(x)$ holds for all $x{\ge}0$. Similarly $\forall\, x > 0$ it holds $\Gamma_{\max}(Q(x)){<}Q(x)$. Fix any positive vector $a > 0$ and consider $\sigma(r){:=}Q(ar){\in}\mathbb{R}^n_+$.

$$\text{Obviously,} \qquad \Gamma_{\max}(\sigma(r)) < \sigma(r),\ \forall r > 0 \tag{23}$$

and by the definition of Q it follows that $\sigma_i \in \mathcal{K}_\infty$ for all $i = 1, \dots, n$. Furthermore, σ_i satisfy:
(i) $\sigma_i^{-1}{\in}Lip_{loc}$ on $(0,\infty)$ (as γ_{ij} is smooth on $(0,\infty)$);
(ii) for every compact set $K \subset (0,\infty)$ there are finite constants $0 < K_1 < K_2$ such that for all points of differentiability of σ_i^{-1} we have

$$0 < K_1 \le (\sigma_i^{-1})'(r) \le K_2, \quad \forall r \in K. \tag{24}$$

In particular, (i) implies that V defined in (19) is locally Lipschitz continuous on $(0,\infty)$ and (ii) implies the bounded growth of σ_i and σ_i^{-1} outside the origin.
Let us show that such function V satisfies (20)-(22).
To this end we define
$\psi_1(|x|) := \min_{i=1,\dots,n} \sigma_i^{-1}(\psi_{i1}(c_1|x|))$ and $\psi_2(|x|) := \max_{i=1,\dots,n} \sigma_i^{-1}(\psi_{i2}(c_2|x|))$ for some suitable positive constants c_1, c_2 that depend on the norm $|\cdot|$. For example, if $|\cdot|$ denotes the infinity norm, then one can take $c_1 = c_2 = 1$. By this choice the condition (20) is satisfied. Define the gain of the whole system by

$$\gamma(|u|) := \max_j \{\phi^{-1}(\gamma_j(|u|))\} \tag{25}$$

with $\phi{\in}\mathcal{K}_\infty$ such that $\phi(t){\le}\max\{\max_{j,j\neq i}\gamma_{ij}(\sigma_j(t)), \sigma_i(t)\}$ for all $t \ge 0$. Using (23) we obtain for each i

$$\max\{\max_{j,j\neq i}\gamma_{ij}(\sigma_j(r)), \phi(r)\} \le \sigma_i(r), \forall r > 0. \tag{26}$$

Consider any $x \neq 0$, as the case $x = 0$ is obvious. Define

$$I{:=}\{i{\in}\{1,\dots,n\}{:}\sigma_i^{-1}(V_i(x_i)){\ge}\max_{j,j\neq i}\sigma_j^{-1}(V_j(x_j))\} \tag{27}$$

i.e. the set of indices i for which the maximum in (19) is attained.
Fix any $i \in I$. If $V(x) \ge \gamma(|u|)$, then by (25) $\phi(V(x)) \ge \gamma_i(|u|)$ and from (26), (27) we have

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) \ &\ge\ \max\{\max_{j,j\neq i}\gamma_{ij}(\sigma_j(V(x))), \phi(V(x))\} \\ &\ge\ \max\{\max_{j,j\neq i}\gamma_{ij}(V_j(x_j)), \gamma_i(|u|)\}. \end{aligned}$$

To show (21) assume $(x,u) \in C$. As V is obtained through the maximization (19), by [4, p.83] we have that

$$\partial V(x) \subset \operatorname{conv}\left\{\bigcup_{i\in I}\partial[\sigma_i^{-1}\circ V_i \circ P_i](x)\right\}, \tag{28}$$

where $P_i(x) := x_i$. Thus we can use the properties of σ_i and V_i to find a bound for $\langle\zeta, f(x,u)\rangle$, $\zeta \in \partial V$. In particular, by the chain rule for Lipschitz continuous functions in [4, Theorem 2.5], we have

$$\partial(\sigma_i^{-1}{\circ}V_i)(x_i){\subset}\{c\zeta_i{:}c{\in}\partial\sigma_i^{-1}(V_i(x_i)),\zeta_i{\in}\partial V_i(x_i)\}, \tag{29}$$

where c is bounded away from zero due to (24). Applying (12) we obtain for all $\zeta_i \in \partial V_i(x_i)$ that

$$\langle\zeta_i, f_i(x,u)\rangle \le -\alpha_i(V_i(x_i)). \tag{30}$$

To get a bound independent on i on the right-hand side of (30) define for $\rho{>}0$, $\widetilde{\alpha}_i(\rho){:=}c_{\rho,i}\alpha_i(\rho){>}0$, where the constant $c_{\rho,i}{:=}K_1$ with K_1 corresponding to the set $K{:=}\{x_i{\in}\chi_i{:}\rho/2{\le}|x_i|{\le}2\rho\}$ given by (24). And for $r{>}0$ define $\hat{\alpha}(r){:=}\min\{\widetilde{\alpha}_i(V_i(x_i))|\,|x|{=}r, V(x) = \sigma_i^{-1}(V_i(x_i))\}{>}0$. Thus using (29)-(30) we obtain

$$\langle\zeta_i, f_i(x,u)\rangle \le -\hat{\alpha}(|x|) \quad \forall\zeta_i{\in}\partial[\sigma_i^{-1}{\circ}V_i](x_i). \tag{31}$$

The same argument applies for all $i{\in}I$. Let us now return to $\zeta{\in}\partial V(x)$. From (28) for any $\zeta{\in}\partial V(x)$ we have that $\zeta{=}\sum_{i\in I}\mu_i c_i\zeta_i$ for suitable $\mu_i{\ge}0, \sum_{i\in I}\mu_i{=}1$, and with $\zeta_i{\in}\partial(V_i{\circ}P_i)(x)$ and $c_i{\in}\partial\sigma_i^{-1}(V_i(x_i))$. Using (31), that $\langle\zeta_i, f(x,u)\rangle{=}\langle P_i(\zeta_i), f_i(x,u)\rangle$ for $\zeta_i{\in}\partial(V_i{\circ}P_i)(x)$ due to the properties of the projection function P_i and that $c_i{>}0$ due to (24), it follows that

$$\begin{aligned} \langle\zeta, f(x,u)\rangle &{=}\sum_{i\in I}\mu_i\langle c_i\zeta_i, f(x,u)\rangle{=}\sum_{i\in I}\mu_i\langle c_iP_i(\zeta_i), f_i(x,u)\rangle \\ &\le -\sum_{i\in I}\mu_i\hat{\alpha}(|x|) \le -\hat{\alpha}(|x|) \le -\hat{\alpha}\circ\psi_2^{-1}\circ V(x). \end{aligned}$$

Thus condition (21) is satisfied with $\alpha := \hat{\alpha}\circ\psi_2^{-1}$.
To show (22) assume now that $(x,u) \in D$. Define

$$\lambda(t) := \max_{i,j,i\neq j}\{\sigma_i^{-1}\circ\gamma_{ij}\circ\sigma_j(t), \sigma_i^{-1}\circ\lambda_i\circ\sigma_i(t)\},\ t > 0. \tag{32}$$

Note that $\sigma_i^{-1}\circ\gamma_{ij}\circ\sigma_j(t) < \sigma_i^{-1}\circ\sigma_i(t) = t$ from (23) and $\sigma_i^{-1}\circ\lambda_i\circ\sigma_i(t) < \sigma_i^{-1}\circ\sigma_i(t) = t$ for all $t > 0$ as $\lambda_i(t){<}t$. Thus $\lambda(t){<}t$, $\forall\, t{>}0$. Let us show that such λ satisfies (22). Condition (13) for ISS-Lyapunov function of subsystem i, the jump behaviour (6) and the assumption $D_i = D$, $i = \{1,\dots,n\}$ imply for $(x,u) \in D$

$$\begin{aligned} V(g(x,u)) &{=}\max_i\sigma_i^{-1}{\circ}V_i(g_i(x,u)) \\ &\le \max_{i,j,i\neq j}\{\sigma_i^{-1}{\circ}\lambda_i(V_i(x_i)), \sigma_i^{-1}{\circ}\gamma_{ij}(V_j(x_j)), \sigma_i^{-1}{\circ}\gamma_i(|u|)\} \\ &= \max_{i,j,i\neq j}\{\sigma_i^{-1}\circ\lambda_i\circ\sigma_i\circ\sigma_i^{-1}(V_i(x_i)), \\ &\qquad \sigma_i^{-1}{\circ}\gamma_{ij}{\circ}\sigma_j{\circ}\sigma_j^{-1}(V_j(x_j)), \sigma_i^{-1}{\circ}\gamma_i(|u|)\} \\ &\le \max\{\lambda(V(x)), \gamma(|u|). \end{aligned}$$

Thus (22) is also satisfied and hence V is an ISS-Lyapunov function of the network (5). □

4 Conclusions

We have shown that a large scale interconnection of ISS hybrid systems is again ISS if the small gain condition is satisfied. The results are provided in terms of trajectories and Lyapunov functions. Moreover an explicit construction of an ISS-Lyapunov function is given. These results extend the corresponding known theorems from [18] to the case of interconnection of more than two hybrid systems and [16] for general type of hybrid systems. However, our results are restricted to interconnections with a common jump set of subsystems.

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A Equivalent definition of an ISS-Lyapunov function

Here we show equivalence between the definition of an ISS-Lyapunov function used in [3] and Definition 2.8.
Consider a function $W : \chi \to \mathbb{R}_+$, $W \in Lip_{loc}$ that satisfies the following properties for (5)
1) There exist functions $\bar{\psi}_1, \bar{\psi}_2 \in \mathcal{K}_\infty$ such that:

$$\bar{\psi}_1(|x|) \leq W(x) \leq \bar{\psi}_2(|x|) \text{ for any } x \in \chi. \tag{A.1}$$

2) There exist function $\bar{\gamma} \in \mathcal{K}$, continuous, positive definite function $\bar{\alpha}_1$ and function $\bar{\alpha}_2 \in \mathcal{K}_\infty$ such that:

$$|x| \geq \bar{\gamma}(|u|) \Rightarrow \forall \zeta \in \partial W(x) \colon \langle \zeta, f(x,u) \rangle \leq -\bar{\alpha}_1(|x|), (x,u) \in C, \tag{A.2}$$

$$|x| \geq \bar{\gamma}(|u|) \Rightarrow W(g(x,u)) - W(x) \leq -\bar{\alpha}_2(|x|), (x,u) \in D. \tag{A.3}$$

In [3] the conditions (A.1)-(A.3) with $\bar{\alpha}_1 \in \mathcal{K}_\infty$ were used to define an ISS-Lyapunov function for (5) and it was shown that existence of such (smooth) function W implies that (5) is ISS. This proof does not change if $\bar{\alpha}_1$ is continuous and positive definite only.

Proposition A.1 *System* (5) *has an ISS-Lyapunov function* V *satisfying* (20)-(22) *if and only if there exists* $W \in Lip_{loc}$ *satisfying* (A.1)-(A.3).

Proof. "$\Rightarrow$" Let V satisfy (20)-(22). We can always majorize a continuous, positive definite function $\lambda < \mathrm{id}$ from (22) by a function $\rho \in \mathcal{K}_\infty$ such that $\lambda(r) \leq \rho(r) < r$ for $r > 0$, for example, $\rho := \frac{1}{2}(\max_{[0,r]} \lambda + \mathrm{id})$. Then for $(x,u) \in D$ from (22) we have

$$V(g(x,u)) \leq \max\{\lambda(V(x)), \gamma(|u|)\} \leq \max\{\rho(V(x)), \gamma(|u|)\}. \tag{A.4}$$

Define

$$\bar{\gamma}(|u|) := \psi_1^{-1} \circ \rho^{-1} \circ \gamma. \tag{A.5}$$

If $|x| \geq \bar{\gamma}(|u|)$, then $\rho(V(x)) \geq \gamma(|u|)$ and using (A.4)

$$V(g(x,u)) \leq \max\{\rho(V(x)), \gamma(|u|)\} = \rho(V(x)) = V(x) - \widetilde{\alpha}(x)$$

$$\Rightarrow V(g(x,u)) - V(x) \leq -\hat{\alpha}(|x|), \tag{A.6}$$

with $\hat{\alpha}(r) := \min_{|s|=r} \widetilde{\alpha}(s)$ that is a continuous, positive definite function, where $\widetilde{\alpha}(s) := V(s) - \rho(V(s)) \geq 0$. From (A.6) and [15, Lemma 2.8], $\exists \bar{\rho}, \bar{\alpha}_2 \in \mathcal{K}_\infty$ such that $W := \bar{\rho} \circ V$ satisfies (A.3) with $\bar{\gamma}(|u|)$ defined in (A.5). As $V(x)$ satisfies (21), then W satisfies (A.2) with $\bar{\alpha}_1 := \widetilde{\rho} \cdot \alpha$ that is continuous, positive definite function, where $\widetilde{\rho} \in \partial \bar{\rho}(V(x))$.
Thus function W satisfies (A.1)-(A.3) with $\bar{\psi}_i := \bar{\rho} \circ \psi_i$.
"$\Leftarrow$" Assume now that function W satisfies (A.1)-(A.3) and define $V := W$, $\psi_1 := \bar{\psi}_1$ and $\psi_2 := \bar{\psi}_2$. Then condition (20) is satisfied. Let

$$\gamma(|u|) := \bar{\psi}_2 \circ \bar{\gamma}(|u|). \tag{A.7}$$

Consider $V(x) \geq \gamma(|u|)$. Then from (A.7), (A.1) $|x| \geq \bar{\gamma}(|u|)$. From (A.1)-(A.2) for all $(x,u) \in C$

$$\forall \zeta \in \partial V(x) \colon \langle \zeta, f(x,u) \rangle \leq -\bar{\alpha}_1(|x|) \leq -\bar{\alpha}_1 \circ \bar{\psi}_2^{-1}(V(x)).$$

Thus V satisfies (21) with $\alpha := \bar{\alpha}_1 \circ \bar{\psi}_2^{-1}$.
By [14, Lemma B.1] for any $\bar{\alpha}_2 \circ \bar{\psi}_2^{-1} \in \mathcal{K}_\infty$ $\exists \widetilde{\alpha} \in \mathcal{K}_\infty$ such that $\widetilde{\alpha} \leq \bar{\alpha}_2 \circ \bar{\psi}_2^{-1}$ and $\mathrm{id} - \widetilde{\alpha} \in \mathcal{K}$. For $V(x) > \gamma(|u|)$ from (A.1) and (A.3)

$$\begin{aligned} V(g(x,u)) &\leq V(x) - \bar{\alpha}_2(|x|) \leq V(x) - \bar{\alpha}_2 \circ \bar{\psi}_2^{-1}(V(x)) \\ &\leq V(x) - \widetilde{\alpha}(V(x)) = (\mathrm{id} - \widetilde{\alpha})(V(x)) = \lambda(V(x)), \end{aligned}$$

where $\lambda := \mathrm{id} - \widetilde{\alpha}$.
Consider now $(x,u) \in D$ such that $V(x) \leq \gamma(|u|)$ and define $\mathcal{A}(|u|) := \{(x,u) \in D : V(x) \leq \gamma(|u|)\}$.
Let us take now $\hat{\gamma}(|u|) := \max_{(x,u) \in \mathcal{A}(|u|)} V(g(x,u))$. Then $V(g(x,u)) \leq \hat{\gamma}(|u|)$. Note that $\hat{\gamma}(0) = 0$ as $V(x) \geq 0 = \gamma(0)$. Furthermore, as function V is nonnegative and $V \in Lip_{loc}$ and function g is continuous, function $\hat{\gamma} \in Lip_{loc}$ is nonnegative. We can always majorize such function $\hat{\gamma}$ by a function $\check{\gamma} \in \mathcal{K}$ such that $\hat{\gamma} \leq \check{\gamma}$. Thus for $(x,u) \in D$ we obtain that $V(g(x,u)) \leq \max\{\check{\gamma}(|u|), \lambda(V(x))\}$ and condition (22) is satisfied with $V := W$ and $\widetilde{\gamma} := \max\{\check{\gamma}, \gamma\}$. $\square$

B Proofs of Theorem 3.1 and Theorem 3.2

We need first the following auxiliary lemmas.

Lemma B.1 *[20, Theorem 2.5.4.] Let the operator* $\Gamma_{\max}$ *defined in* (14) *satisfy* (15). *Then there exists* $\phi \in \mathcal{K}_\infty$ *s.t. for all* $w, v \in \mathbb{R}^n_+$,

$$w \leq \max\{\Gamma_{\max}(w), v\} := \begin{pmatrix} \max\{\max_{j, j \neq 1} \gamma_{1j}(w_j), v_1\} \\ \vdots \\ \max\{\max_{j, j \neq n} \gamma_{nj}(w_j), v_n\} \end{pmatrix}$$

implies $\|w\| \leq \phi(\|v\|)$.

Lemma B.2 *Let* $s : \operatorname{dom} s \to \mathbb{R}^n_+$ *be continuous between the jumps and bounded with unbounded* $\operatorname{dom} s$. *Then*

$$\limsup_{(t,k) \in \operatorname{dom} s, t+k \to \infty} s(t,k) = \limsup_{t+k \to \infty} \|s_{[(t/2, k/2), \lim_{\tau + j \to \infty} (\tau, j)]}\|.$$

Proof. The proof goes along the lines of the proof of a similar result for continuous systems in Lemma 3.2 in [5] but instead of time t we consider the points (t,k) of the time domain. $\square$

B.1 Proof of Theorem 3.1

Let us take the supremum over $(\tau, l) \leq (t,k)$ on both sides of (8)

$$\begin{aligned} \|x_{i(t,k)}\|_{(\bar{\tau}, \bar{l})} \leq \max\{&\sigma_i(|x_{i0}|), \\ &\max_{j, j \neq i} \hat{\gamma}_{ij}(\|x_{j(t,k)}\|_{(\bar{\tau}, \bar{l})}), \hat{\gamma}_i(\|u\|_{(\bar{\tau}, \bar{l})})\}, \end{aligned} \tag{B.1}$$

where $(\bar{\tau}, \bar{l}) := \max_{(\tau, l) \in \operatorname{dom} x} (\tau, l)$, i.e. the maximum element of $\operatorname{dom} x$.
Let $\Gamma := (\hat{\gamma}_{ij})_{n \times n}$, $w := \left(\|x_{1(t,k)}\|_{(\bar{\tau}, \bar{l})}, \ldots, \|x_{n(t,k)}\|_{(\bar{\tau}, \bar{l})} \right)^T$,

$$
\begin{aligned}
v &:= \begin{pmatrix} \max\{\sigma_1(|x_{10}|), \hat{\gamma}_1(\|u\|_{(\bar{\tau},\bar{l})})\} \\ \vdots \\ \max\{\sigma_n(|x_{n0}|), \hat{\gamma}_n(\|u\|_{(\bar{\tau},\bar{l})})\} \end{pmatrix} \\
&= \max\{\sigma(|x_0|), \hat{\gamma}(\|u\|_{(\bar{\tau},\bar{l})})\},
\end{aligned}
$$

where $\sigma, \hat{\gamma} \in \mathcal{K}_\infty$. From (B.1) we obtain $w \leq \max\{\Gamma_{\max}(w), v\}$. Then by Lemma B.1 $\exists\, \phi \in \mathcal{K}_\infty$ s.t.

$$
\begin{aligned}
|x(t,k)| &\leq \|x_{(t,k)}\|_{(\bar{\tau},\bar{l})} \leq \phi(\|\max\{\sigma(|x_0|), \hat{\gamma}(\|u\|_{(\bar{\tau},\bar{l})})\}\|) \\
&\leq \max\{\phi(\|\sigma(|x_0|)\|), \phi(\|\hat{\gamma}(\|u\|_{(\bar{\tau},\bar{l})})\|)\}.
\end{aligned}
\tag{B.2}
$$

for all $(t,k) \in \operatorname{dom} x$. Hence for every initial condition and essentially bounded input u the solution of the system (5) exists and is bounded, since the right-hand side of (B.2) does not depend on t, k. From the last line in (B.2) the estimate (16) for the pre-GS follows. □

B.2 Proof of Theorem 3.2

Let (τ, l) be an arbitrary initial point of the time domain. From the definition of the AG property we have

$$
\begin{aligned}
&\limsup_{(t,k)\in \operatorname{dom} x_i, t+k\to\infty} |x_i(t,k)| \\
&\leq \max\{\max_{j, j\neq i} \widetilde{\gamma}_{ij}(\|x_{j[(\tau,l), \lim_{\bar{\tau}+\bar{l}\to\infty}(\bar{\tau},\bar{l})]}\|_\infty), \widetilde{\gamma}_i(\|u\|_\infty)\}.
\end{aligned}
$$

Then from Lemma 3.6 in [3] it follows that

$$
\begin{aligned}
&\limsup_{(t,k)\in \operatorname{dom} x_i, t+k\to\infty} |x_i(t,k)| \\
&\leq \max\{\max_{j, j\neq i} \widetilde{\gamma}_{ij}(\limsup_{\tau+l\to\infty}(\|x_{j[(\tau,l), \lim_{\bar{\tau}+\bar{l}\to\infty}(\bar{\tau},\bar{l})]}\|_\infty)) \widetilde{\gamma}_i(\|u\|_\infty)\}.
\end{aligned}
\tag{B.3}
$$

Since all solutions of (1) are bounded the following holds by Lemma B.2:

$$
\limsup_{\substack{(t,k)\in \operatorname{dom} x_i, \\ t+k\to\infty}} |x_i(t,k)| = \limsup_{\tau+l\to\infty}(\|x_{i[(\tau,l), \lim_{\bar{\tau}+\bar{l}\to\infty}(\bar{\tau},\bar{l})]}\|_\infty) =: l_i(x_i).
$$

By this property from (B.3) follows

$$
l_i(x_i) \leq \max\{\max_{j, j\neq i} \widetilde{\gamma}_{ij}(l_j(x_j)), \widetilde{\gamma}_i(\|u\|_\infty)\}.
$$

Using Lemma B.1 for $\Gamma = (\widetilde{\gamma}_{ij})_{n\times n}$, $w_i = l_i(x_i)$ and $v_i := \widetilde{\gamma}_i(\|u\|_\infty)$ we conclude

$$
\limsup_{(t,k)\in \operatorname{dom} x, t+k\to\infty} |x(t,k)| \leq \phi(\|u\|_\infty) \tag{B.4}
$$

for some $\phi \in \mathcal{K}$, which is the desired AG property. □

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